
Parameter-Invariant Monitor Design for Cyber-Physical Systems: Supplemental Slides

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Example 1 : LRT

$$\mathcal{H}_0 : (\theta_1, \theta_2) \in \{1\} \times \{0\}$$

$$\mathcal{H}_1 : (\theta_1, \theta_2) \in \{0\} \times \{1\}$$

apply the likelihood ratio test

distribution of measurements

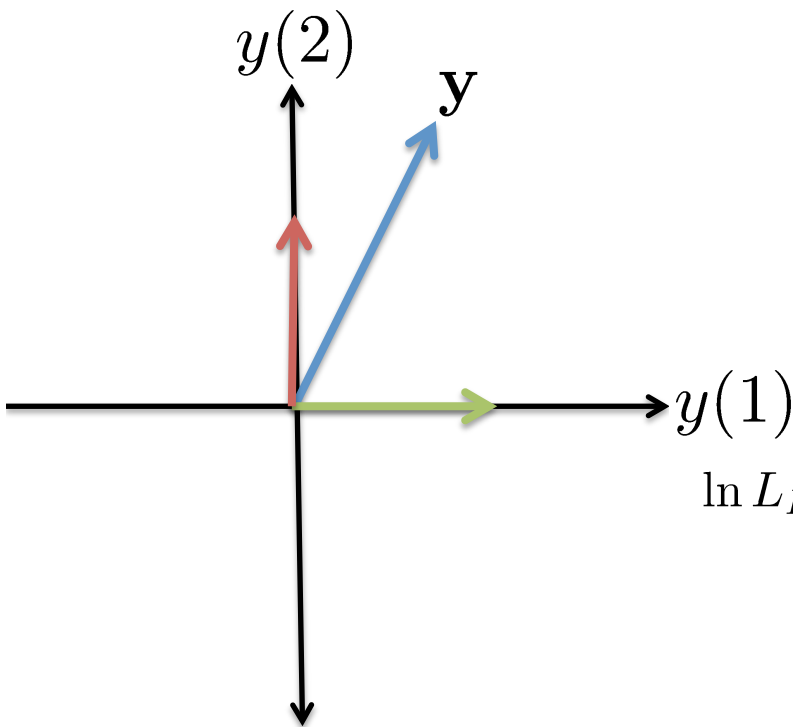
$$f(\mathbf{y}|\theta) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \sum_{k=1}^2 (y(k) - \theta_k)^2 \right\}$$

log-likelihood of parameters

$$\ln L(\theta; \mathbf{y}) = -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \sum_{k=1}^2 (y(k) - \theta_k)^2$$

log-likelihood ratio

$$\begin{aligned} \ln L_R(\mathbf{y}) &= \ln L(\Theta_1; \mathbf{y}) - \ln L(\Theta_0; \mathbf{y}) \\ &= \frac{1}{2} \left(\sum_{k=1}^2 (y(k) - \Theta_{0,k})^2 - \sum_{k=1}^2 (y(k) - \Theta_{1,k})^2 \right) \\ &= \frac{1}{2} ((y(1) - 1)^2 + y(2)^2 - y(1)^2 - (y(2) - 1)^2) \\ &= y(2) - y(1) \end{aligned}$$



Example 2 : GLRT

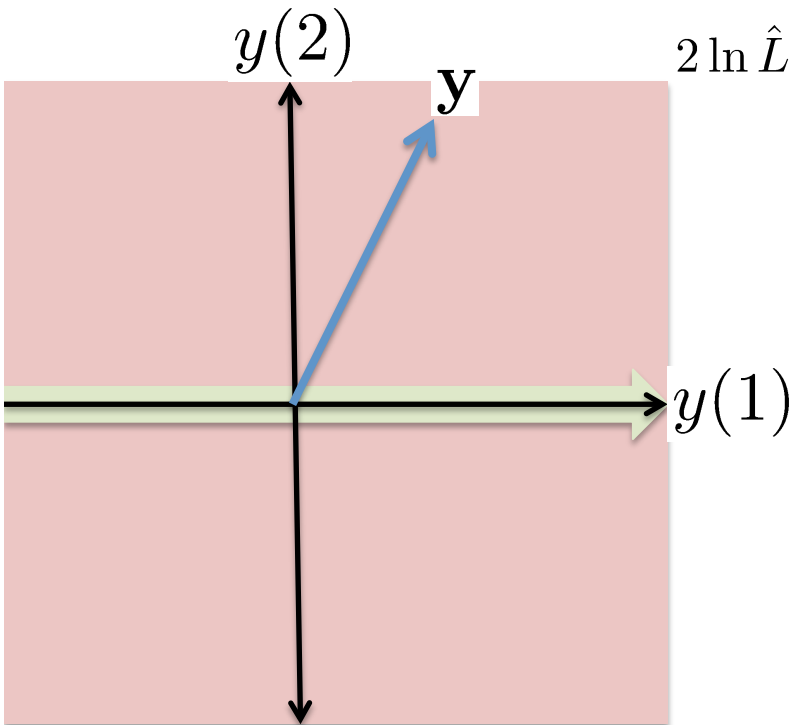
$$\mathcal{H}_0 : (\theta_1, \theta_2) \in \mathbb{R} \times \{0\}$$

$$\mathcal{H}_1 : (\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}$$

apply the generalized likelihood ratio test

generalized log-likelihood ratio

$$\begin{aligned} 2 \ln \hat{L}_R(\mathbf{y}) &= \max_{\hat{\theta}_1 \in \Theta_1} 2 \ln L(\hat{\theta}_1; \mathbf{y}) - \max_{\hat{\theta}_0 \in \Theta_0} 2 \ln L(\hat{\theta}_0; \mathbf{y}) \\ &= \min_{\hat{\theta}_0 \in \mathbb{R} \times \{0\}} \sum_{k=1}^2 (y(k) - \hat{\theta}_{0,k})^2 \\ &\quad - \min_{\hat{\theta}_1 \in \mathbb{R} \times \mathbb{R}} \sum_{k=1}^2 (y(k) - \hat{\theta}_{1,k})^2 \\ &= y(2)^2 \end{aligned}$$



Example 3 : GLRT

$$\mathcal{H}_0 : (\theta_1, \theta_2) \in \mathbb{R} \times \{0\}$$

$$\mathcal{H}_1 : (\theta_1, \theta_2) \in \{0\} \times \mathbb{R}$$

apply the generalized likelihood ratio test

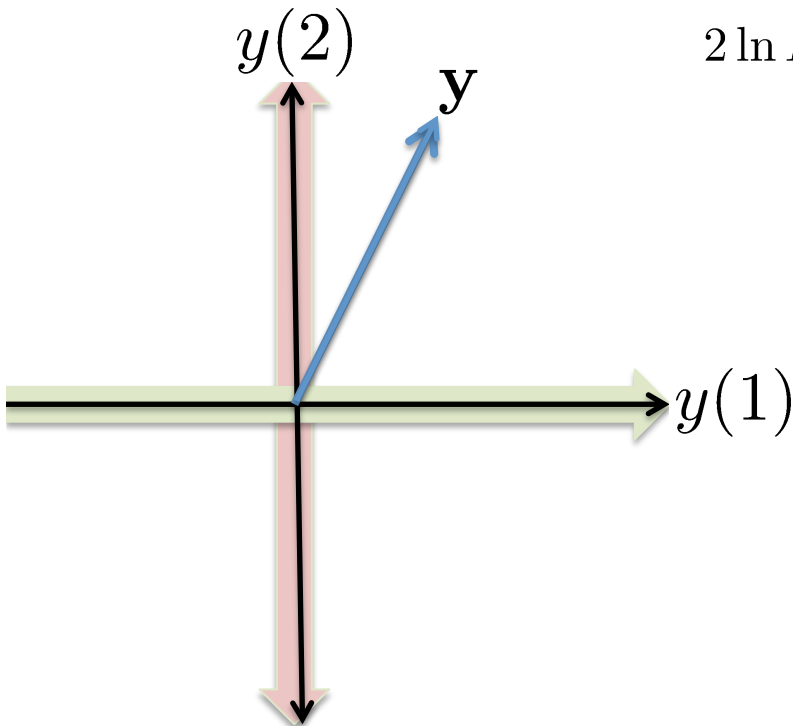
generalized log-likelihood ratio

$$2 \ln \hat{L}_R(\mathbf{y}) = \max_{\hat{\theta}_1 \in \Theta_1} 2 \ln L(\hat{\theta}_1; \mathbf{y}) - \max_{\hat{\theta}_0 \in \Theta_0} 2 \ln L(\hat{\theta}_0; \mathbf{y})$$

$$= \min_{\hat{\theta}_0 \in \mathbb{R} \times \{0\}} \sum_{k=1}^2 (y(k) - \hat{\theta}_{0,k})^2$$

$$- \min_{\hat{\theta}_1 \in \{0\} \times \mathbb{R}} \sum_{k=1}^2 (y(k) - \hat{\theta}_{1,k})^2$$

$$= y(2)^2 - y(1)^2$$



Example 2 : MI test

$$\mathcal{H}_0 : (\theta_1, \theta_2) \in \mathbb{R} \times \{0\}$$

$$\mathcal{H}_1 : (\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}$$

apply maximally invariant test

generalized log likelihood over MI statistic

$$\begin{aligned} 2 \ln \hat{L}_R(t(\mathbf{y})) &= \max_{c \in \mathbb{R}} 2 \ln L(c; t(\mathbf{y})) - 2 \ln L(0; t(\mathbf{y})) \\ &= y(2)^2 - \min_{c \in \mathbb{R}} (y(2) - c)^2 \\ &= y(2)^2 \end{aligned}$$

