Robust Medical Monitor Design:
Part 2 – Design of Parameter-Invariant Monitors

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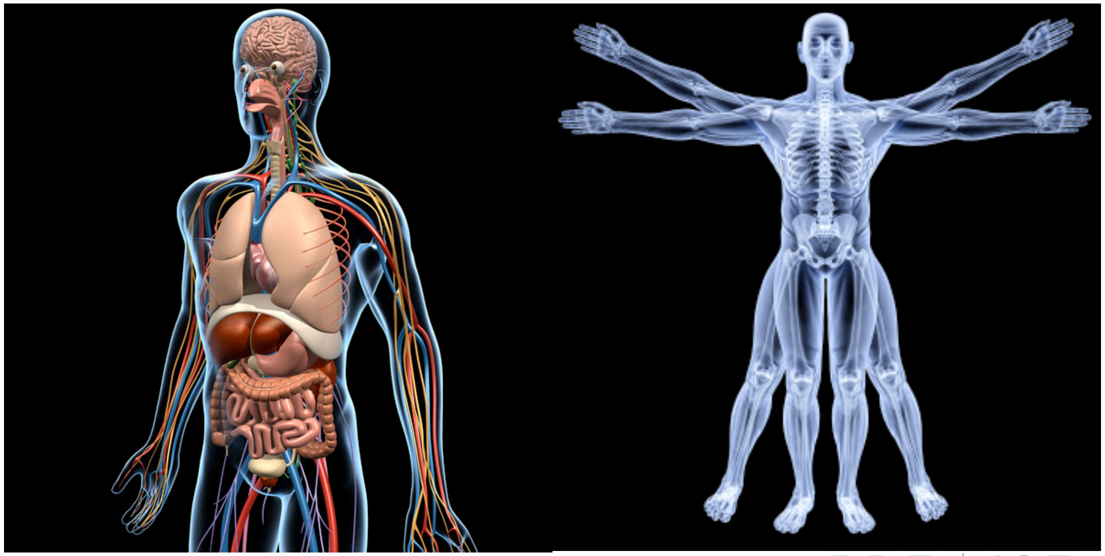
Recall Health Monitoring



Recall The Monitor Design Problem

- Design a binary test between:
  - H_0 : null hypothesis
  - H_1 : event hypothesis

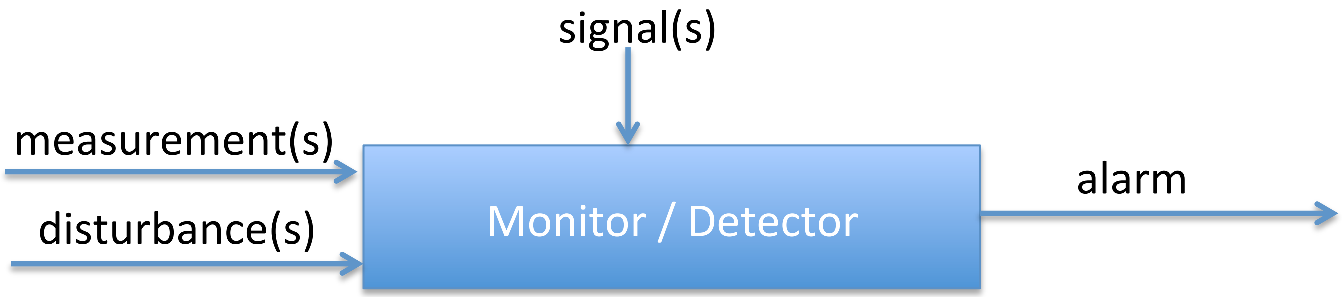
	H_0 is true	H_1 is true
test claims H_0	correct non-detection	missed detection
test claims H_1	false positive	true positive

- Performance constraints:
  - bound false positive rate
  - maximize true positive rate

- Last module covered different tests:
  - likelihood ratio test (LRT)
  - generalized likelihood ratio test (GLRT)
  - maximally invariant (MI) tests
  - parameter invariant (PAIN) tests

- This module covers the design of parameter-invariant monitors
  - modeling monitoring events
  - invariance to real-world dynamics

Monitor Architecture



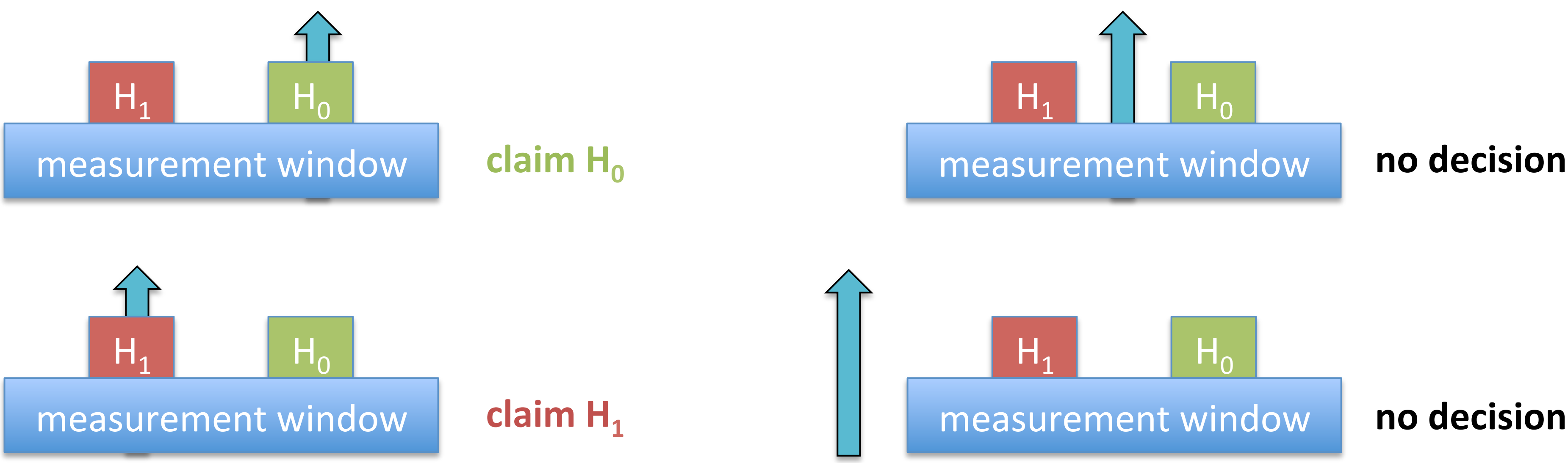
- measurement(s) – gathered from real-world sensor(s)
  - affected by monitoring event
- signal(s) – provided by sensors or domain expertise
  - unique for each hypothesis (ideally)
  - example: meal detection type I diabetes using constant glucose monitoring
    - H_0 : meal happens at time X
    - H_1 : meal does not happen at time X (… meal happens at time Y…)
- disturbances(s) - provided by sensors or domain expertise
  - weakly correlated with monitoring event
  - potentially correlated with measurements
  - examples:
    - insulin bolus – Diabetic meal detection using glucose measurements
    - respiratory rate – Hypoxia detection using expired CO_2

What is an "event"

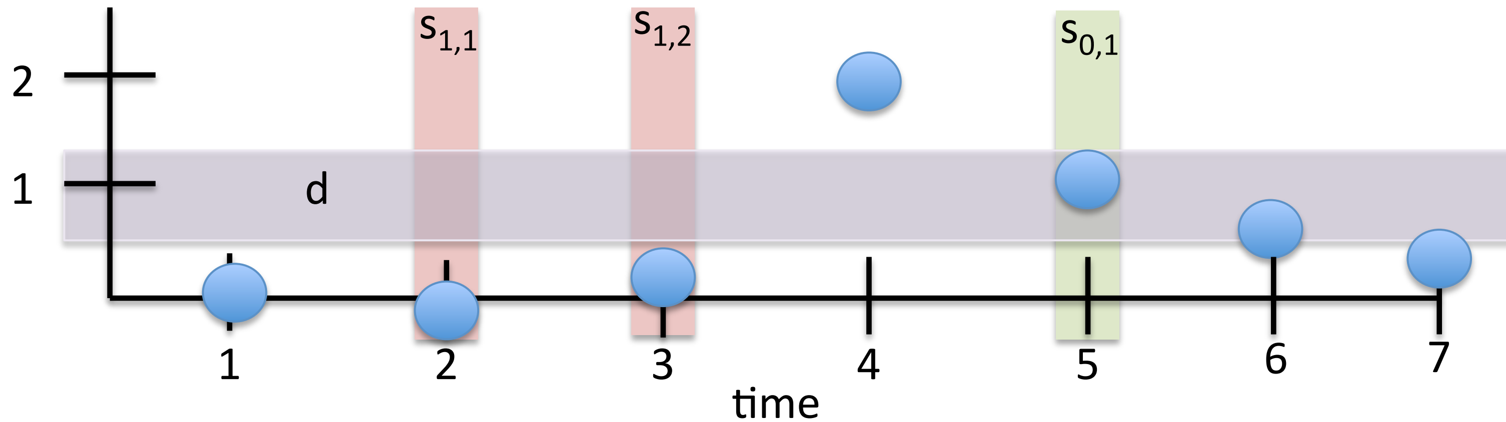
Modeling Monitoring Events

- Many ways to model "an event": (not necessarily mutually exclusive)
  - presence/absence events
    - e.g. Does an input affect the output?
  - sequential events
    - e.g. In which time frame did the event occur?
  - dynamic events
    - e.g. Did the dynamics change?

rich paradigm



Sequential Monitoring Example



measurements

$$[y(1),\ y(2),\ y(3),\ y(4),\ y(5),\ y(6)\ y(7)]^\top = [0.0,\ -0.1,\ 0.1,\ 1.9,\ 1.0,\ 0.6,\ 0.2]^\top$$

disturbance

$$d(k) = d$$

null signal

$$s_{0,1}(k) = s_{0,1}\delta(k-5)$$

event signals

$$s_{1,1}(k) = s_{1,1}\delta(k-2)$$
$$s_{1,2}(k) = s_{1,2}\delta(k-3)$$

d, $s_{0,1}$, $s_{1,1}$, $s_{1,2}$ are unknown

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How do the inputs affect the measurements???

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Finite-time Modeling of Physical Systems

- Consecutive temporal samples of inputs and measurements
  - inputs : test and/or nuisance
  - outputs (measurements) : dependent on inputs (and dynamics)

- Three classes of input-output dynamic models for physical systems
  - White box : model is known
  - Grey box : model is partially known
  - Black box : model is unknown

- What is the input-output relation?
  - linear / non-linear
  - static / dynamic
  - discrete / continuous dynamics
  - probabilistic / deterministic
  - etc.

Over a finite window of time, (practically) all input-output relations approximate one another with bounded error

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Linear Time Invariant Models

- Lets consider a linear time invariant model:
  - Easier to analyze than non-linear and time-varying models
    - well developed theory
    - many practical applications exist

$$y(k) = \sum_{n=1}^{N} \boxed{a_n} y(k-n) + \sum_{m=1}^{M} \boxed{b_{m,n}} u_m(k-n) + \boxed{\sigma n(k)}$$ (general form)

known

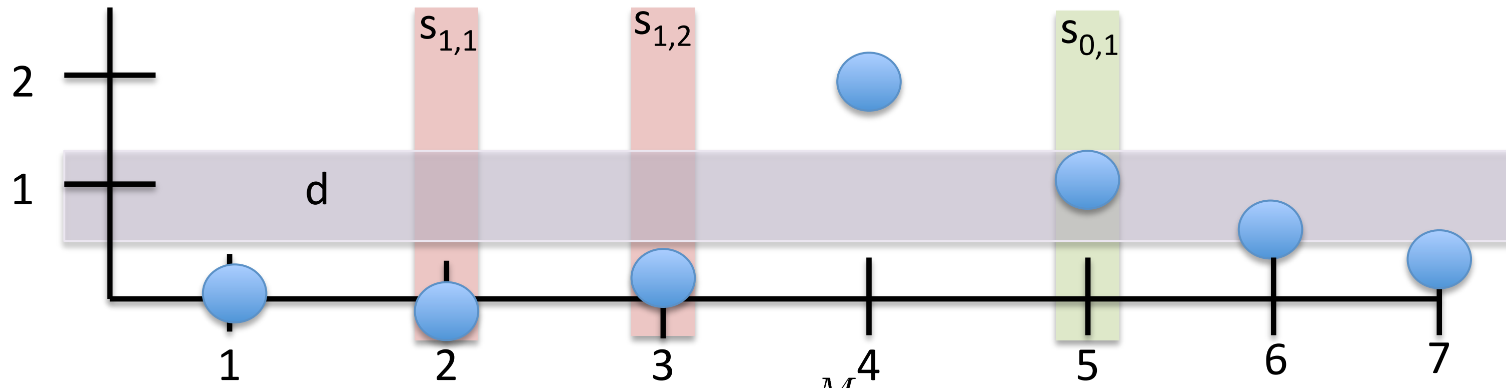
- model order: N
- number of inputs: M
- measurement: $y(k)$
- inputs: $u_m(k)$
  - test signals and disturbances
- noise mean: $E[n(k)] = 0$

unknown

- dynamics: $a_n, b_{m,n}$
- noise: $n(k)$
- noise scale: σ

let noise represent the model error

Sequential Monitoring Example



Model order: N = 1 → $y(k) = a_1 y(k-1) + \sum_{m=1}^{M} b_{m,1} u_m(k-1) + \sigma n(k)$

Define inputs as concatenated test signals and disturbances: (4 inputs)

$$\left[\begin{array}{cccc} u_1(k) & u_2(k) & u_3(k) & u_4(k) \end{array}\right] = \left[\begin{array}{cccc} s_{0,1}(k) & s_{1,1}(k) & s_{1,2}(k) & d(k) \end{array}\right]$$

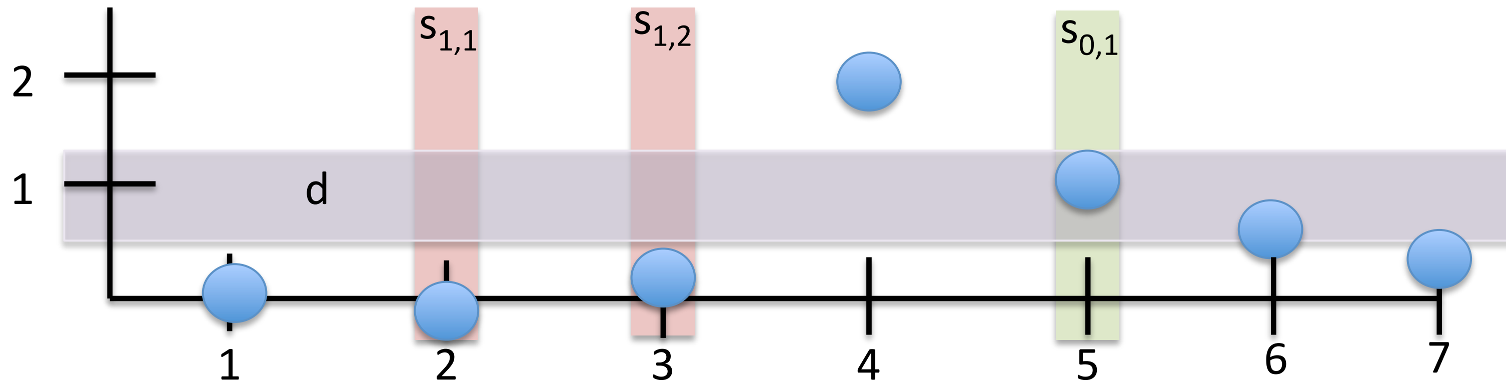
Resulting input-output relationships:

$$y(2) = a_1 y(1) + b_{4,1} + \sigma n(2)$$
$$y(3) = a_1 y(2) + b_{2,1} + b_{4,1} + \sigma n(3)$$
$$y(4) = a_1 y(3) + b_{3,1} + b_{4,1} + \sigma n(4)$$
$$y(5) = a_1 y(4) + b_{4,1} + \sigma n(5)$$
$$y(6) = a_1 y(5) + b_{1,1} + b_{4,1} + \sigma n(6)$$
$$y(7) = a_1 y(6) + b_{4,1} + \sigma n(7)$$

$$\begin{bmatrix} -0.1 \\ 0.1 \\ 1.9 \\ 1.0 \\ 0.6 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.0 & & & & 1 \\ -0.1 & & 1 & & 1 \\ 0.1 & & & 1 & 1 \\ 1.9 & & & & 1 \\ 1.0 & 1 & & & 1 \\ 0.6 & & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \end{bmatrix} + \sigma \begin{bmatrix} n(2) \\ n(3) \\ n(4) \\ n(5) \\ n(6) \\ n(7) \end{bmatrix}$$

time concatenated model

Sequential Monitoring Example



time concatenated model

$$\begin{bmatrix} -0.1 \\ 0.1 \\ 1.9 \\ 1.0 \\ 0.6 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.0 & & & & 1 \\ -0.1 & & 1 & & 1 \\ 0.1 & & & 1 & 1 \\ 1.9 & & & & 1 \\ 1.0 & 1 & & & 1 \\ 0.6 & & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \end{bmatrix} + \sigma \begin{bmatrix} n(2) \\ n(3) \\ n(4) \\ n(5) \\ n(6) \\ n(7) \end{bmatrix}$$

$$\begin{bmatrix} u_1(k) & u_2(k) & u_3(k) & u_4(k) \end{bmatrix} = \begin{bmatrix} s_{0,1}(k) & s_{1,1}(k) & s_{1,2}(k) & d(k) \end{bmatrix}$$

null hypothesis → $b_{2,1} = 0$, $b_{3,1} = 0$

$$\begin{bmatrix} -0.1 \\ 0.1 \\ 1.9 \\ 1.0 \\ 0.6 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.0 & & 1 \\ -0.1 & & 1 \\ 0.1 & & 1 \\ 1.9 & & 1 \\ 1.0 & 1 & 1 \\ 0.6 & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_{1,1} \\ b_{4,1} \end{bmatrix} + \sigma \begin{bmatrix} n(2) \\ n(3) \\ n(4) \\ n(5) \\ n(6) \\ n(7) \end{bmatrix}$$

$$\mathcal{H}_0 \ : \ \mathbf{y} = \mathbf{H}_0 \theta_0 + \sigma \mathbf{n}$$

event hypothesis → $b_{1,1} = 0$

$$\begin{bmatrix} -0.1 \\ 0.1 \\ 1.9 \\ 1.0 \\ 0.6 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.0 & & & 1 \\ -0.1 & 1 & & 1 \\ 0.1 & & 1 & 1 \\ 1.9 & & & 1 \\ 1.0 & & & 1 \\ 0.6 & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \end{bmatrix} + \sigma \begin{bmatrix} n(2) \\ n(3) \\ n(4) \\ n(5) \\ n(6) \\ n(7) \end{bmatrix}$$

$$\mathcal{H}_1 \ : \ \mathbf{y} = \mathbf{H}_1 \theta_1 + \sigma \mathbf{n}$$

Linear Time Invariant Model for PAIN monitoring

- General LTI model form: $y(k) = \sum_{n=1}^{N} a_n y(k-n) + \sum_{m=1}^{M} b_{m,n} u_m(k-n) + \sigma n(k)$

- Window of T+N+1 measurements:

$$\begin{bmatrix} y(k-T) \\ \vdots \\ y(k-1) \\ y(k) \end{bmatrix} = \begin{bmatrix} y(k-T-1) & \dots & y(k-T-N) \\ \vdots & & \vdots \\ y(k-2) & \dots & y(k-N-1) \\ y(k-1) & \dots & y(k-N) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} + \sum_{m=1}^{M} \begin{bmatrix} u_m(k-T-1) & \dots & u_m(k-T-N) \\ \vdots & & \vdots \\ u_m(k-2) & \dots & u(k-N-1) \\ u_m(k-1) & \dots & u(k-N) \end{bmatrix} \begin{bmatrix} b_{m,1} \\ \vdots \\ b_{m,N} \end{bmatrix} + \sigma \begin{bmatrix} n(k-T) \\ \vdots \\ n(k) \end{bmatrix}$$

$$\mathbf{y} = \mathbf{G}_0 \mathbf{a} + \sum^{M} \mathbf{G}_m \mathbf{b}_m + \sigma \mathbf{n}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \dots & \mathbf{G}_M \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b}_1 \\ . \end{bmatrix} + \sigma \mathbf{n}$$

Potential nuisance transformations: translation, scale, rotation

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$$\mathcal{H}_j \ : \ \mathbf{y} = \mathbf{H}_j \theta_j + \sigma \mathbf{n}, \quad j \in \{0, 1\}$$

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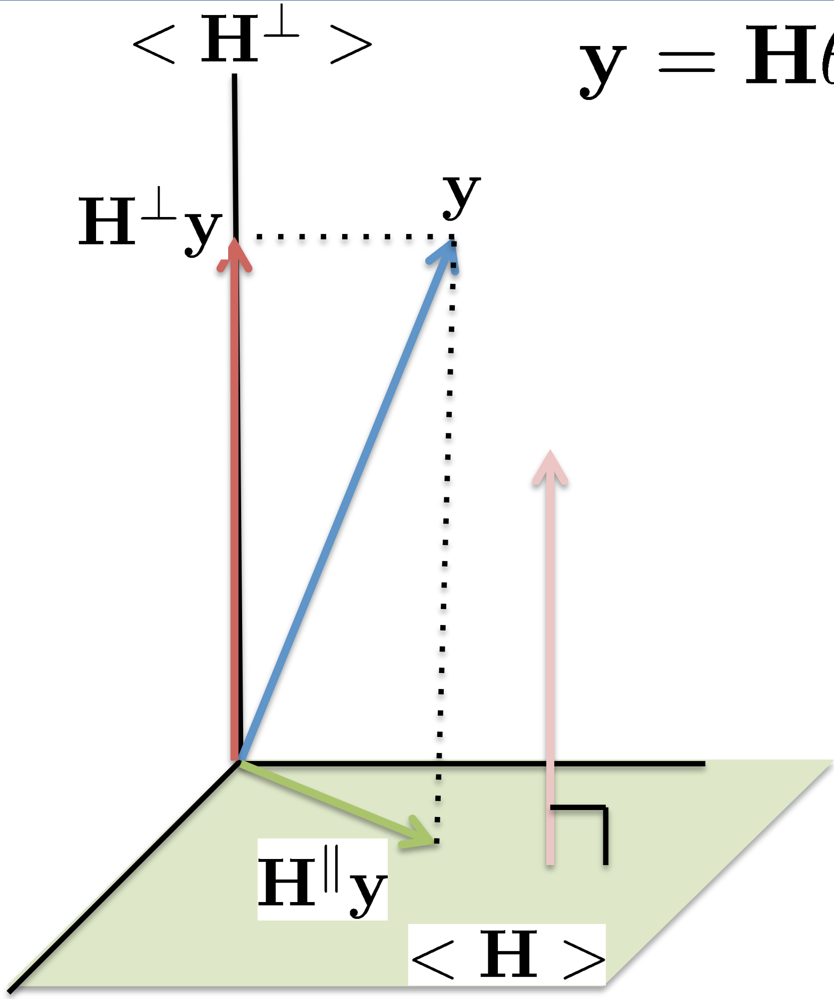
Nuisance Transformation

$$\mathbf{y} = \mathbf{H}\theta + \sigma\mathbf{n}$$



space of H : $< \mathbf{H} >$

projection onto $< \mathbf{H} >$: $\mathbf{H}^{\|}$

nullspace H : $< \mathbf{H}^{\perp} >$

projection onto $< \mathbf{H}^{\perp} >$: $\mathbf{H}^{\perp}$

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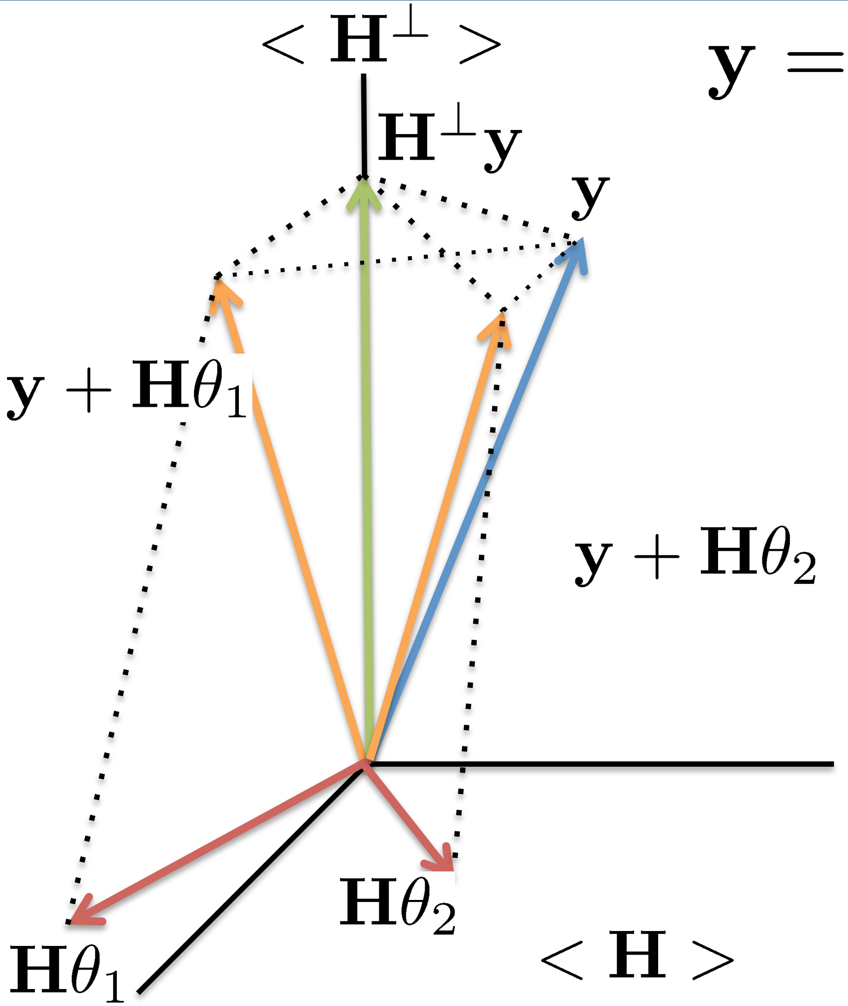
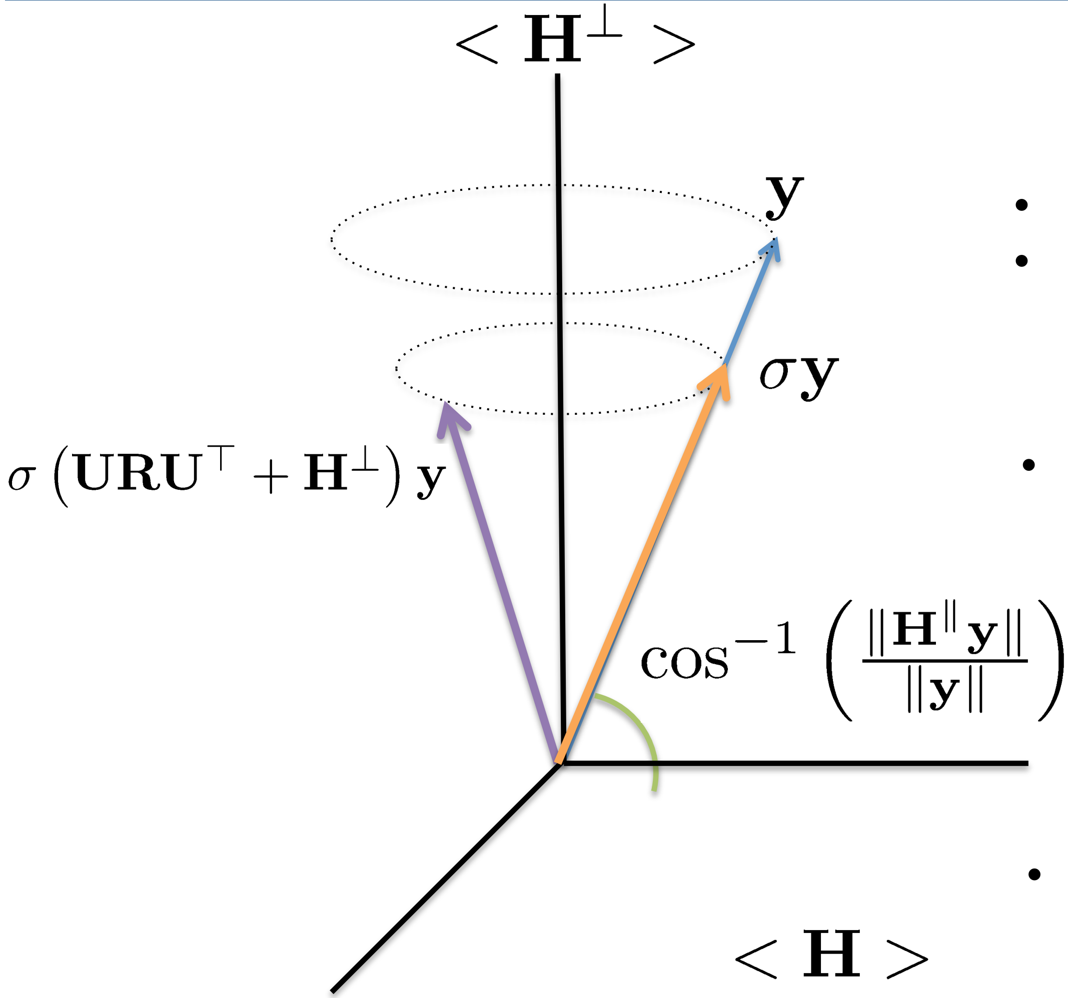
Nuisance Transformation : Translation



$$\mathbf{y} = \mathbf{H}\theta + \sigma\mathbf{n}$$

- **translation induced by parameters corresponding to nuisance inputs**
- **maximally invariant statistic :** $\mathbf{H}^{\perp}\mathbf{y}$
  - $\mathbf{H}^{\perp}\mathbf{H} = \mathbf{0}$

Nuisance Transformation : Scale and Rotation



$$\mathbf{y} = \mathbf{H}\theta + \sigma\mathbf{n}$$

- scale induced by unknown model error
- rotation induced when events can happen at multiple time steps
  - "don't care when, just if it happens"
- mathematical relationships:

$$\mathbf{UU}^{\top} = \mathbf{H}^{\|}$$

$$\mathbf{RR}^{\top} = \mathbf{I}$$

- maximally invariant statistic : $\frac{\|\mathbf{H}^{\|}\mathbf{y}\|}{\|\mathbf{y}\|}$

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Parameter Invariant Test

hypothesis testing problem

$$\mathcal{H}_0 : y = \mathbf{H}_0\theta_0 + \sigma\mathbf{n}$$

$$\mathcal{H}_1 : y = \mathbf{H}_1\theta_1 + \sigma\mathbf{n}$$

- maximally invariant statistics:
  - translation : $\mathbf{H}^{\perp}\mathbf{y}$
  - rotation and scale : $\frac{\|\mathbf{H}^{\|}\mathbf{y}\|}{\|\mathbf{y}\|}$

"composition of maximally invariant statistics is maximally invariant"

- Parameter Invariant statistics:
  - **constant false positive rate:**

$$\hat{\mathbf{y}}_0 = \mathbf{H}_0^{\perp}\mathbf{y}, \quad \hat{\mathbf{H}}_0 = \mathbf{H}_0^{\perp}\mathbf{H}_1, \quad t_0 = \frac{\|\hat{\mathbf{H}}_0^{\|}\hat{\mathbf{y}}_0\|}{\|\hat{\mathbf{y}}_0\|}$$

  - **constant true positive rate:**

$$\hat{\mathbf{y}}_1 = \mathbf{H}_1^{\perp}\mathbf{y}, \quad \hat{\mathbf{H}}_1 = \mathbf{H}_1^{\perp}\mathbf{H}_0, \quad t_1 = \frac{\|\hat{\mathbf{H}}_1^{\|}\hat{\mathbf{y}}_1\|}{\|\hat{\mathbf{y}}_1\|}$$

- 2-sided parameter invariant test:

$$\phi(t_0, t_1) = \begin{cases} 1, & t_0 > \eta_0 \ \wedge \ t_1 \leq \eta_1 \\ 0, & t_0 \leq \eta_0 \ \wedge \ t_1 > \eta_1 \\ ?, & \text{else} \end{cases}$$

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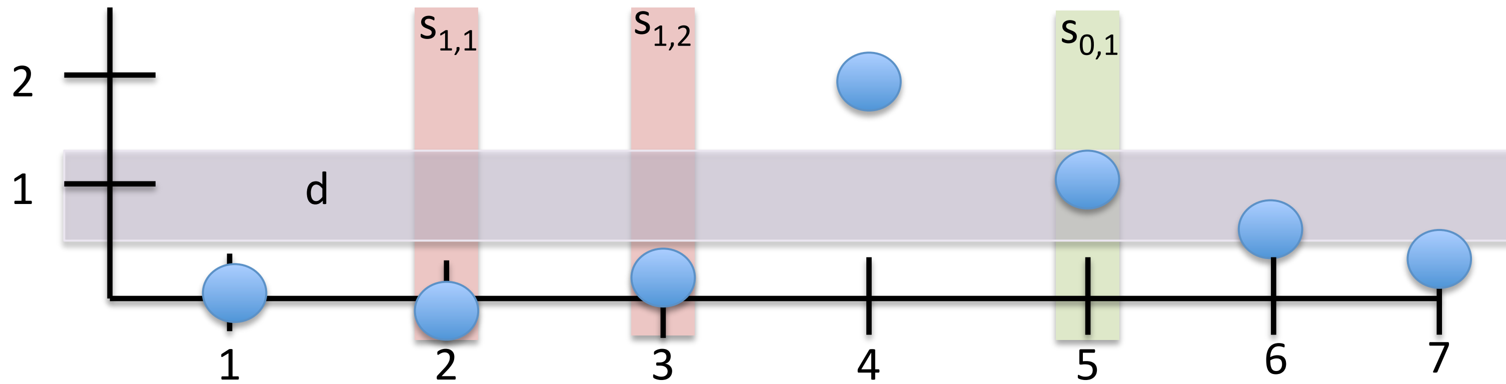
Sequential Monitoring Example



$$\mathcal{H}_0 \ : \ \mathbf{y} = \mathbf{H}_0\theta_0 + \sigma\mathbf{n}$$

$$\mathcal{H}_1 \ : \ \mathbf{y} = \mathbf{H}_1\theta_1 + \sigma\mathbf{n}$$

$$\mathbf{y} = \begin{bmatrix} -0.1 \\ 0.1 \\ 1.9 \\ 1.0 \\ 0.6 \\ 0.2 \end{bmatrix} \quad \mathbf{H}_0 = \begin{bmatrix} 0.0 & & 1 \\ -0.1 & & 1 \\ 0.1 & & 1 \\ 1.9 & & 1 \\ 1.0 & 1 & 1 \\ 0.6 & & 1 \end{bmatrix} \quad \mathbf{H}_1 = \begin{bmatrix} 0.0 & & & 1 \\ -0.1 & 1 & & 1 \\ 0.1 & & 1 & 1 \\ 1.9 & & & 1 \\ 1.0 & & & 1 \\ 0.6 & & & 1 \end{bmatrix}$$

matlab code

$$\hat{\mathbf{H}}_0 = \text{null}\left(\mathbf{H}_0'\right)' * \mathbf{H}_1 \approx \begin{bmatrix} 0 & 0.59 & 0.27 & 0 \\ 0 & -0.57 & 0.77 & 0 \\ 0 & 0.04 & 0.29 & 0 \end{bmatrix}$$

$$\hat{\mathbf{y}}_0 = \text{null}\left(\mathbf{H}_0'\right)' * \mathbf{y} \approx \begin{bmatrix} 0.68 \\ 1.34 \\ -0.59 \end{bmatrix}$$

$$t_0 = \ \text{norm}\left(\text{orth}(\hat{\mathbf{H}}_0)' * \hat{\mathbf{y}}_0\right) / \text{norm}\left(\hat{\mathbf{y}}_0\right) \approx \ 0.9997 \qquad t_1 \approx 0.9515$$

$\cos^{-1}(t_0)$ ~ 1 degree
$\cos^{-1}(t_1)$ ~ 18 degrees

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Module Summary

- Given any hypothesis testing problem of the form:

$$\mathcal{H}_0 \;:\; \mathbf{y} = \mathbf{H}_0\theta_0 + \sigma\mathbf{n}$$
$$\mathcal{H}_1 \;:\; \mathbf{y} = \mathbf{H}_1\theta_1 + \sigma\mathbf{n}$$

  - the 2-sided PAIN test presented herein generates a robust monitor.
  - sequential events is just 1 approach that results in this form

- The following module explores the application of the 2-sided PAIN test herein
  - meal-detection for type I diabetics
  - early detection of hypoxia in infants

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